JUMPING CONICS ON A SMOOTH QUADRIC IN $\mathbb{P}_3$

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ABSTRACT. We investigate the jumping conics of stable vector bundles E of rank 2 on a smooth quadric surface Q with the first Chern class $c_1 = \mathcal{O}_Q(-1,-1)$ with respect to the ample line bundle $\mathcal{O}_Q(1,1)$. We show that the set of jumping conics of E is a hypersurface of degree $c_2(E)-1$ in $\mathbb{P}_3^*$. Using these hypersurfaces, we describe moduli spaces of stable vector bundles of rank 2 on Q in the cases of lower $c_2(E)$.

1. Introduction

The moduli space of stable sheaves on surfaces has been studied by many people. Especially, over the projective plane, the moduli space of stable sheaves of rank 2 was studied by W.Barth [1] and K.Hulek [9], using the jumping lines and jumping lines of the second kind. In [15], this idea was generalized to the jumping conics on the projective plane. In this article, we use the concept of jumping conics on the smooth quadric surface.

Let Q be a smooth quadric in $\mathbb{P}_3 = \mathbb{P}(V)$, where V is a 4-dimensional vector space over complex numbers $\mathbb{C}$, and $\mathcal{M}(k)$ be the moduli space of stable vector bundles of rank 2 on Q with the Chern classes $c_1 = \mathcal{O}_Q(-1,-1)$ and $c_2 = k$ with respect to the ample line bundle $H = \mathcal{O}_Q(1,1)$. $\mathcal{M}(k)$ form an open Zariski subset of the projective variety $\overline{\mathcal{M}}(k)$ whose points correspond to the semi-stable sheaves on Q with the same numerical invariants. The Zariski tangent space of $\mathcal{M}(k)$ at E, is naturally isomorphic to $H^1(Q, \mathrm{End}(E))$ and so the dimension of $\mathcal{M}(k)$ is equal to $h^1(Q,\mathrm{End}(E)) = 4k-5$, since E is simple.

Using the Beilinson-type theorem on Q [3], we obtain the following monad for $E \in \mathcal{M}(k)$,

$$0 \to \mathbb{C}^{k-1}\otimes\mathcal{O}_Q(-1,-1) \to \mathbb{C}^k\otimes(\mathcal{O}_Q(0,-1)\oplus\mathcal{O}_Q(-1,0)) \to \mathbb{C}^{k-1}\otimes\mathcal{O}_Q \to 0,$$

with the cohomology sheaf E, where the first injective map derives a map

$$\delta : H^1(E(-1,-1)) \otimes V^* \to H^1(E).$$

As in [2], we similarly define $S(E) \subset \mathbb{P}_3^*$, the set of jumping conics of E, and prove that $S(E)$ is a hypersurface in $\mathbb{P}_3^*$ of degree $k-1$ whose equation is

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given by $\det \delta(z) = 0$, $z \in V^*$, where $\delta(z)$ is a symmetric $(k-1) \times (k-1)$-matrix. We give a criterion for $H \in \mathbb{P}_3^*$ to be a singular point of $S(E)$ and calculate the exact number of singular points of $S(E)$ when E is a Hulsbergen bundle, i.e. E admits the following exact sequence,

$$0 \to \mathcal{O}_Q \to E(1,1) \to I_Z(1,1) \to 0,$$

where Z is a 0-cycle on Q with length k whose support is in general position.

In Section 4, we describe the above results in the cases $c_2 \le 3$ by investigating the map

$$S : \mathcal{M}(k) \to |\mathcal{O}_{\mathbb{P}_3^*}(k-1)|,$$

sending E to $S(E)$. When $c_2 = 2$, $S(E)$ is a hypersurface in $\mathbb{P}_3$ and $\mathcal{M}(2)$ is isomorphic to $\mathbb{P}_3 \backslash Q$ via S, which was already shown in [8]. In the case of $c_2 = 3$, we investigate the surjective map from $\mathcal{M}(3)$ to $\mathbb{P}_3^*$, sending E to the vertex point of the quadric cone $S(E) \subset \mathbb{P}_3^*$ to give an explicit description of $\mathcal{M}(3)$. In fact, the generic fibre of this map over $H \in \mathbb{P}_3^*$ is isomorphic to the set of smooth conics which are Poncelet related to the smooth conic $H \cap Q$. As a result, we can observe that S is an isomorphism from $\mathcal{M}(3)$ to its image and in particular, when $c_2 = 2, 3$, the set of jumping conics, $S(E)$, determines E uniquely.

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2. The Beilinson Theorem and Jumping Conics

2.1. **The Beilinson Theorem.** Let V_1 and V_2 be two 2-dimensional vector spaces with the coordinate $[x_{1i}]$ and $[x_{2j}]$, respectively. Let Q be a smooth quadric isomorphic to $\mathbb{P}(V_1) \times \mathbb{P}(V_2)$ and then it is embedded into $\mathbb{P}_3 \simeq \mathbb{P}(V)$ by the Segre map, where $V = V_1 \otimes V_2$. Let us denote $f^*\mathcal{O}_{\mathbb{P}_1}(a) \otimes g^*\mathcal{O}_{\mathbb{P}_1}(b)$ by $\mathcal{O}_Q(a,b)$ and $E \otimes \mathcal{O}_Q(a,b)$ by $E(a,b)$ for coherent sheaves E on Q, where f and g are the projections from Q to each factors. Then the canonical line bundle K_Q of Q is $\mathcal{O}_Q(-2,-2)$.

Definition 2.1. For a fixed ample line bundle H on Q, a torsion free sheaf E of rank r on Q is called *stable* (resp. *semi-stable*) with respect to H if

$$\frac{\chi(F \otimes \mathcal{O}_Q(mH))}{r'} < (resp. \le) \frac{\chi(E \otimes \mathcal{O}_Q(mH))}{r},$$

for all non-zero subsheaves $F \subset E$ of rank r'.

Let $\overline{\mathcal{M}}(k)$ be the moduli space of semi-stable sheaves of rank 2 on Q with the Chern classes $c_1 = \mathcal{O}_Q(-1,-1)$ and $c_2 = k$ with respect to the ample line bundle $H = \mathcal{O}_Q(1,1)$. The existence and the projectivity of $\overline{\mathcal{M}}(k)$ is known in [6] and it has an open Zariski subset $\mathcal{M}(k)$ which consists of the stable vector bundles with the given numeric invariants. By the Bogomolov theorem, $\mathcal{M}(k)$ is empty if $4k < c_1^2 = 2$ and in particular, we can consider

only the case $k \geq 1$. Note that $E \simeq E^*(-1,-1)$ and by the Riemann-Roch theorem, we have

$$\chi_E(m) := \chi(E(m,m)) = 2m^2 + 2m + 1 - k,$$

for $E \in \overline{\mathcal{M}}(k)$.

Using the same trick as in the proof of the Beilinson theorem on the vector bundles over the projective space [13], we can obtain similar statement over Q.

Proposition 2.2. [3] *For any holomorphic bundle E on Q, there is a spectral sequence*

$$E_1^{p,q} \Rightarrow E_\infty^{p+q} = \begin{cases} E, & \text{if } p+q=0; \\ 0, & \text{otherwise,} \end{cases}$$

with

$$\begin{cases} E_1^{p,q} = 0, \ |p+1| > 1 \\ E_1^{0,q} = H^q(E) \otimes \mathcal{O}_Q \\ E_1^{-2,q} = H^q(E(-1,-1)) \otimes \mathcal{O}_Q(-1,-1), \end{cases}$$

and an exact sequence

$$\cdots \to H^q(E(0,-1)) \otimes \mathcal{O}_Q(0,-1) \to E_1^{-1,q} \to H^q(E(-1,0)) \otimes \mathcal{O}_Q(-1,0) \to \cdots .$$

Proof. Let p_1 and p_2 be the projections from $Q \times Q$ to each factors and denote $p_1^*\mathcal{O}_Q(a,b) \otimes p_2^*\mathcal{O}_Q(c,d)$ by $\mathcal{O}(a,b)(c,d)'$. If we let $\triangle$ be the diagonal of $Q \times Q$, we have the following Koszul complex,

$$(1) \quad 0 \to \mathcal{O}(-1,-1)(-1,-1)' \to \bigoplus_{i=0}^{1} \mathcal{O}(-i,1-i)(-i,1-i)' \to \mathcal{O} \to \mathcal{O}_\triangle .$$

If we tensor it with p_2^*E, then we have a locally free resolution of $p_2^*E|_\triangle$. If we take higher direct images under p_1, we get the assertion by the standard argument on the spectral sequence. □

From the stability condition of $E \in \mathcal{M}(k)$, we have $H^0(E(a,b)) = 0$ whenever $a+b \leq 0$. Hence $E_1^{p,q} = 0$ for $p = -2,-1,0$ and $q = 0,2$ and thus the proposition gives us a monad

$$(2) \qquad M : 0 \to K_{1,1} \otimes \mathcal{O}_Q(-1,-1) \to E_1^{-1,1} \to K_{0,0} \otimes \mathcal{O}_Q \to 0,$$

with the cohomology sheaf $E(M) = E$, where $K_{a,b} = H^1(E(-a,-b))$ and $E_1^{-1,1}$ fits into the following exact sequence,

$$(3) \qquad 0 \to K_{0,1} \otimes \mathcal{O}_Q(0,-1) \to E_1^{-1,1} \to K_{1,0} \otimes \mathcal{O}_Q(-1,0) \to 0.$$

Since $H^1(\mathcal{O}_Q(1,-1)) = 0$, this exact sequence splits. Thus we have the following corollary.

Corollary 2.3. *Let $E \in \mathcal{M}(k)$. Then E becomes the cohomology sheaf of the following monad:*

$$M(E) \ : \ 0 \to K_{1,1} \otimes \mathcal{O}_Q(-1,-1) \to \bigoplus_{i=0}^{1} (K_{i,1-i} \otimes \mathcal{O}_Q(-i,-1+i)) \to K_{0,0} \otimes \mathcal{O}_Q \to 0.$$

Note that $k_{1,1} = k_{0,0} = k-1$ and $k_{1,0} = k_{0,1} = k$, where $k_{i,j} = \dim K_{a,b}$.

Let us denote by a, the first injective map in the monad in the corollary (2.3). Since $E \simeq E^*(-1,-1)$, the last surjective map is the dual of a, twisted by $\mathcal{O}_Q(-1,-1)$ and thus the monad M is completely determined by a. The monomorphism a corresponds to an element α in

$$K_{1,1}^* \otimes ((K_{0,1} \otimes V_1) \oplus (K_{1,0} \otimes V_2)),$$

i.e. $\alpha = (\alpha_1, \alpha_2)$, where $\alpha_i \in \mathrm{Hom}(V_i^*, \mathrm{Hom}(K_{1,1}, K_{i-1,2-i}))$. Since $K_{1,1}^* \simeq K_{0,0}$ and $K_{1,0}^* \simeq K_{0,1}$, we can obtain a map

$$\delta : V_1^* \otimes V_2^* \to \mathrm{Hom}(K_{1,1}, K_{0,0}), \tag{4}$$

defined by $\delta := \alpha_2^t \circ \alpha_1 + \alpha_1^t \circ a_2$. So $\delta(z) \in K_{0,0} \otimes K_{0,0}$ since $K_{1,1}^* \simeq K_{0,0}$. Again from the self-duality of E, i.e. $E \simeq E^*(-1,-1)$, we have $\delta(z) = \delta(z)^t$. In other words, $\delta(z)$ is an element in $\mathrm{Sym}^2(K_{0,0})$ for all z.

2.2. **Jumping Conics.** Let H be a general hyperplane section of $\mathbb{P}_3$ and then $C_H := Q \cap H$ is a conic on H. Let E be a vector bundle of rank r on Q. If we choose an isomorphism $f : \mathbb{P}_1 \to C_H$, then due to Grothendieck, we have

$$f^* E|_{C_H} \simeq \mathcal{O}_{\mathbb{P}_1}(a_1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}_1}(a_r),$$

where $a_{E,H} := (a_1, \cdots, a_r) \in \mathbb{Z}^r$ such that $a_1 \geq \cdots \geq a_r$. Here, $a_{E,H}$ is called the splitting type of $E|_{C_H}$.

Definition 2.4. A conic $C_H = Q \cap H$ on Q, is called *a jumping conic* of E if the splitting type $a_{E,H}$ of $E|_{C_H}$ is different from the generic splitting type a_E. We will denote the set of jumping conics of E by $S(E) \subset \mathbb{P}_3^*$.

Remark 2.5. The above definition is valid only for the general hyperplane sections H. Later, we give an equivalent definition for the jumping conics for arbitrary case, using the cohomological criterion.

From the theorem (0.2) in [12], we have $a_i - a_{i+1} \leq 2$ for all i since the degree of $Q \subset \mathbb{P}_3$ is 2. From the following proposition, we know that this upper bound can be sharpened to be 1.

Proposition 2.6. *If E is a stable vector bundle on Q of rank r, we have*

$$a_i - a_{i+1} \leq 1, \ \textit{for all } i,$$

where $a_E = (a_1, \cdots, a_r)$.

Proof. The proof is analogue of the one for the Grauert-Mülich theorem in [13]. We consider the incidence variety $\mathbf{I} = \{(x,H) \in Q \times \mathbb{P}_3^* \mid x \in C_H\}$ with the projections π_1 and π_2 to each factors. Suppose that i is the first index such that $a_i - a_{i+1} \geq 2$. Moreover, we can assume that $a_i = 0$. Now, let us consider the natural map

$$\pi_2^* \pi_{2*} \pi_1^* E \to \pi_1^* E \tag{5}$$

and E_1 be the image of this map. Then E_1 is a subsheaf of $\pi_1^*(E)$ on $\mathbf{I}$ of rank i such that

$$f^* E_1|_{\pi_2^{-1}(H)} \simeq \mathcal{O}_{\mathbb{P}_1}(a_1) \oplus \cdots \mathcal{O}_{\mathbb{P}_1}(a_i),$$

for a general $H \in \mathbb{P}_3^*$ and an isomorphism $f : \mathbb{P}_1 \to C_H$. Then the quotient sheaf $E_2 := \pi_1^*(E)/E_1$ is of rank $r - i$ with

$$f^* E_2|_{\pi_2^{-1}(H)} \simeq \mathcal{O}_{\mathbb{P}_1}(a_{i+1}) \oplus \cdots \mathcal{O}_{\mathbb{P}_1}(a_r),$$

for a general $H \in \mathbb{P}_3^*$. From the following lemma, the pull-back of the relative tangent bundle $T_{\mathbf{I}|Q}$ to $\mathbb{P}_1$, is $\mathcal{O}_{\mathbb{P}_1}(-1)^{\oplus 2}$. Hence, the restriction of the sheaf $\mathcal{H}om(T_{\mathbf{I}|Q}, \mathcal{H}om(E_1, E_2))$ to C_H is isomorphic to the direct sum of $\mathcal{O}_{\mathbb{P}_1}(a_{j_1} - a_{j_2} + 1)^{\oplus 2}$ where $j_1 \geq i+1$ and $j_2 \leq i$. In particular, we have

$$\operatorname{Hom}(T_{\mathbf{I}|Q}, \mathcal{H}om(E_1, E_2)) = 0.$$

By the Descente-Lemma [13], there exists a subsheaf of E' of E on Q such that $\pi_1^* E' = E_1$ and it would make the contradiction to the stability of E. □

Lemma 2.7. *Let $f : \mathbb{P}_1 \to C_H$ be an isomorphism. Then we have am isomorphism*

$$f^* T_{\mathbf{I}|Q}|_{C_H} \simeq \mathcal{O}_{\mathbb{P}_1}(-1)^{\oplus 2}.$$

Proof. Let $\mathbf{I}'$ be the incidence variety in $\mathbb{P}_3 \times \mathbb{P}_3^*$, i.e. $\mathbf{I}' \simeq \mathbb{P}(T_{\mathbb{P}_3}^*)$. Then we have the universal exact sequence of $\mathbb{P}(T_{\mathbb{P}_3}^*)$,

$$0 \to M \to \pi_1^* T_{\mathbb{P}_3}^* \to N \to 0,$$

where M and N are vector bundles on $\mathbf{I}'$ of rank 1 and 2, respectively. If we restrict the universal sequence to $\pi_2^{-1}(\{H\})$, then we obtain

$$0 \to N_{H|\mathbb{P}_3}^* \to T_{\mathbb{P}_3}^*|_H \to T_H^* \to 0,$$

where $N_{H|\mathbb{P}_3} \simeq \mathcal{O}_H(1)$ is the normal bundle of H in $\mathbb{P}_3$. Note that

$$T_{\mathbf{I}'|\mathbb{P}_3}|_H \simeq \mathcal{H}om(M, N)|_H \simeq N_{H|\mathbb{P}_3} \otimes T_H^*.$$

Since $f^* N_{H|\mathbb{P}_3}$ is isomorphic to $\mathcal{O}_{\mathbb{P}_1}(2)$, it is enough to prove that

$$f^* T_H^* \simeq f^* \Omega_H \simeq \mathcal{O}_{\mathbb{P}_1}(-3)^{\oplus 2}$$

since $T_{\mathbf{I}|\mathbb{P}_3}$ is the restriction of $T_{\mathbf{I}'|\mathbb{P}_3}$ to $\mathbf{I}$. If we tensor the following exact sequence,

$$0 \to \mathcal{O}_{\mathbb{P}_2}(-2) \to \mathcal{O}_{\mathbb{P}_2} \to \mathcal{O}_{C_H} \to 0,$$

with $\Omega_{\mathbb{P}_2}(1)$ (recall that $H \simeq \mathbb{P}_2$ and C_H is the image conic of $\mathbb{P}_1$ by f) and take the long exact sequence of cohomology, then we obtain $H^0(\Omega_{\mathbb{P}_2}(1)|_{C_H}) = 0$ from the Bott theorem [13]. Thus we have

$$h^0(f^*(\Omega_{\mathbb{P}_2}(1))) = h^0(\Omega_{\mathbb{P}_2}(1)|_{C_H}) = 0.$$

Since $c_1(f^*(\Omega_{\mathbb{P}_2}(1))) = -2$, the only possibility is that $f^*(\Omega_{\mathbb{P}_2}(1)) \simeq \mathcal{O}_{\mathbb{P}_1}(-1)^{\oplus 2}$ and so $f^*T_H^* \simeq \mathcal{O}_{\mathbb{P}_1}(-3)^{\oplus 2}$. □

Remark 2.8. From the theorem 1 in [15], we know that for a semistable vector bundle of rank 2 on $\mathbb{P}_2$ and a general smooth conic $f : \mathbb{P}_1 \hookrightarrow \mathbb{P}_2$, we have

$$f^*E \simeq \begin{cases} \mathcal{O}_{\mathbb{P}_1}^{\oplus 2}, & \text{if } c_1(E) = 0; \\ \mathcal{O}_{\mathbb{P}_1}(-1)^{\oplus 2}, & \text{if } c_1(E) = -1. \end{cases}$$

Since $T_{\mathbb{P}_2}$ is semistable, we can also obtain $f^*T_{\mathbb{P}_2}^* \simeq \mathcal{O}_{\mathbb{P}_1}(-3)^{\oplus 2}$ for a general conic. In fact, this is true for all conics and $T_{\mathbb{P}_2}$ is the only vector bundle of rank 2 on $\mathbb{P}_2$ up to twists with the same splitting type over all smooth conics except a direct sum of two line bundles [11].

Let us assume that $E \in \mathcal{M}(k)$ be a stable vector bundle on Q. As an immediate consequence of the above proposition, we get the following corollary.

Corollary 2.9. *For a general conic C_H on Q, we have*

$$E|_{C_H} \simeq \mathcal{O}_{C_H}(-p) \oplus \mathcal{O}_{C_H}(-p),$$

where p is a point on C_H.

In particular, the jumping conics of E can be characterized by

$$h^0(E|_{C_H}) \neq 0, \tag{6}$$

and we will use this cohomological criterion as the definition of the jumping conics of E.

We consider the exact sequence,

$$0 \to E(-1,-1) \to E \to E|_{C_H} \to 0,$$

to derive the following long exact sequence,

$$0 \to H^0(E|_{C_H}) \to H^1(E(-1,-1)) \to H^1(E), \tag{7}$$

where the last map is given by $\delta(z) = \alpha_2^t \otimes \alpha_1 + \alpha_1^t \otimes \alpha_2$, where z is the coordinates determining the hyperplane section H. Hence C_H is a jumping conic if and only if $\det(\delta) = 0$. Note that $\det(\delta)$ is a homogeneous polynomial of degree $c_2(E) - 1$ with the coordinates of $V_1^* \otimes V_2^*$. This determinant does not vanish identically due to the proposition (2.6), and so we obtain that $S(E)$ is a hypersurface of degree $c_2(E) - 1$ in $\mathbb{P}_3^*$. From the fact that $\delta(z) = \delta(z)^t$, we obtain the following statement.

Theorem 2.10. *$S(E)$ is a symmetric determinantal hypersurface of degree $c_2(E) - 1$ in $\mathbb{P}_3^*$.*

The natural question on $S(E)$ is the smoothness and the next proposition will give an answer to this question.

Proposition 2.11. *If $h^0(E|_{C_H}) \geq 2$, then $H \in \mathbb{P}_3^*$ is a singular point of $S(E)$.*

Proof. The statement is clear from the theory on the singular locus of symmetric determinantal varieties [7]. Indeed, let $M = M_0$ denote the projective space $\mathbb{P}_N$ of $(k-1) \times (k-1)$ matrices up to scalars and M_i be the locus of matrices of corank i or more. Let us consider a map $\varphi : \mathbb{P}_3^* \to M$, determined naturally by δ. If we let S_i be the preimage of M_i via φ, then we have

$$\begin{aligned} T_p S_2 &= d\varphi^{-1}(T_q\varphi(S_2)) \\ &= d\varphi^{-1}(T_q M_2 \cap T_q\varphi(\mathbb{P}_3^*)) \\ &= d\varphi^{-1}(M \cap T_q\varphi(\mathbb{P}_3^*)) \text{ ,since } T_q M_2 = M \text{ [7]} \\ &= d\varphi^{-1} T_q\varphi(\mathbb{P}_3^*) = \mathbb{P}_3^*, \end{aligned}$$

where $q = \varphi(p)$ and $p \in S_2$. In particular, S_2 is the singular locus of $S_1 = S(E)$. $\square$

Remark 2.12. Let $f : \mathbb{P}_1 \to C_H \subset Q$ be a smooth conic on Q and assume that we have

$$f^* E|_{C_H} \simeq \mathcal{O}_{\mathbb{P}_1}(-1-i) \oplus \mathcal{O}_{\mathbb{P}_1}(-1+i),$$

where i is a nonnegative integer. Note that $k = h^0(E|_{C_H}) = \text{corank}(\delta(z))$, where z is the coordinates of H. If $i \geq 2$, then $H \in \mathbb{P}_3^*$ is a singular point of $S(E)$.

Now for later use, let us define a sheaf supported on $S(E)$. As in [1], we can see that $S(E)$ is the support of the $\mathcal{O}_{\mathbb{P}_3^*}$-sheaf $\vartheta_E(1)$ defined by the following exact sequence,

$$0 \to K_{1,1} \otimes \mathcal{O}_{\mathbb{P}_3^*}(-1) \to K_{0,0} \otimes \mathcal{O}_{\mathbb{P}_3^*} \to \vartheta_E(1) \to 0. \tag{8}$$

The first injective map is composed of

$$K_{1,1} \otimes \mathcal{O}_{\mathbb{P}_3^*}(-1) \to K_{1,1} \otimes (V_1^* \otimes V_2^*) \otimes \mathcal{O}_{\mathbb{P}_3^*} \to K_{0,0} \otimes \mathcal{O}_{\mathbb{P}_3^*},$$

where the first map is from the Euler sequence over $\mathbb{P}_3^*$ and the second map is from the map δ. So ϑ_E is an $\mathcal{O}_{S(E)}$-sheaf.

From the incidence variety $\mathbf{I} \subset Q \times \mathbb{P}_3^*$, we obtain

$$0 \to \pi_1^*\mathcal{O}_Q(-1,-1) \otimes \pi_2^*\mathcal{O}_{\mathbb{P}_3^*}(-1) \to \mathcal{O}_{Q \times \mathbb{P}_3^*} \to \mathcal{O}_{\mathbf{I}} \to 0.$$

If we tensor it with $\pi_1^* E$ and take the direct image of it, we obtain,

$$0 \to K_{1,1} \otimes \mathcal{O}_{\mathbb{P}_3^*}(-1) \to K_{0,0} \otimes \mathcal{O}_{\mathbb{P}_3^*} \to R^1\pi_{2*}\pi_1^* E \to 0.$$

Since this exact sequence coincide with the sequence (8), we have

Lemma 2.13. $\vartheta_E(1) \simeq R^1\pi_{2*}\pi_1^* E$.

3. Examples

Let $\mathcal{M}(k)$ be the moduli space of stable vector bundles of rank 2 on Q with the Chern classes $c_1 = \mathcal{O}_Q(-1,-1)$ and $c_2 = k$ with respect to the ample line bundle $\mathcal{O}_Q(1,1)$. The dimension of $\mathcal{M}(k)$ can be computed to be $4k-5$. By sending $E \in \mathcal{M}(k)$ to the set of jumping conics of E, we can define a morphism

$$S : \mathcal{M}(k) \to |\mathcal{O}_{\mathbb{P}_3^*}(k-1)| \simeq \mathbb{P}_N,$$

where $N = \binom{k+2}{3} - 1$.

Let $Z = \{x_1, \cdots, x_k\}$ be a 0-dimensional subscheme of Q with length k in general position. If E is a stable vector bundle fitted into the exact sequence,

$$0 \to \mathcal{O}_Q \to E(1,1) \to I_Z(1,1) \to 0,$$

which is called *a Hulsbergen bundle*, then E is in $\mathcal{M}(k)$. Note that if $k \leq 4$, then $E \in \mathcal{M}(k)$ admits the above exact sequence. Conversely, let us consider the above extension. It is parametrized by

$$\mathbb{P}(Z) := \mathbb{P}\,\mathrm{Ext}^1(I_Z(1,1), \mathcal{O}_Q) \simeq \mathbb{P}H^0(\mathcal{O}_Z)^*.$$

If we give $\mathbb{P}(Z)$ the coordinate system $(c_1, \cdots, c_k)$ corresponding to Z, then by the lemma (5.1.2) in Chapter 1 [13] or [4], the bundle E corresponding to $(c_1, \cdots, c_k)$ is locally free if and only if $c_i \neq 0$ for all i.

Now by the theorem (2.10), $S(E) \subset \mathbb{P}_3^*$ is a hypersurface of degree $k-1$.

Lemma 3.1.

(1) *If* $|Z \cap H| \geq 3$*, then* $h^0(E|_{C_H}) \geq 2$.
(2) *If* $|Z \cap H| \leq 2$*, then* $h^0(E|_{C_H}) \leq 1$.

Proof. Let $m = |Z \cap H| \geq 3$. If C_H is a smooth conic, then by tensoring the above exact sequence with $\mathcal{O}_H$, we have $E|_{C_H} \simeq \mathcal{O}_{C_H}((m-1)p) \oplus \mathcal{O}_{C_H}(-mp)$ since $\mathrm{Ext}^1(\mathcal{O}_{C_H}(-mp), \mathcal{O}_{C_H}((m-1)p)) = 0$. Thus, $h^0(E_{C_H}) \geq 2$.

Let us assume that $C_H = l_1 + l_2$, i.e. H is a tangent plane of Q. Note that

$$h^0(C_H, \mathcal{O}(a_1,a_2)) = \begin{cases} 0, & \text{if } a_i < 0; \\ a_i, & \text{if } a_i \geq 0,\ a_j < 0; \\ a_1 + a_2 + 1, & \text{if } a_i \geq 0 \end{cases}$$

where $\mathcal{O}(a_1,a_2) := \mathcal{O}_{l_1}(a_1) \cup \mathcal{O}_{l_2}(a_2)$. From the lemma (2.1) in [11], it is clear that $h^0(E_{C_H}) \geq 2$. For example, when $m = 3$ and $Z \cap H = \{x,y,z\}$, $x, y \in l_1$, $z \in l_2$ and $q = l_1 \cap l_2 \notin Z$, we have

$$0 \to \mathcal{O}_{l_1}(1) \cup \mathcal{O}_{l_2} \to E \to \mathcal{O}_{l_1}(-2) \cup \mathcal{O}_{l_2}(-1) \to 0, \tag{9}$$

and in particular the filtrations in the lemma (2.1) of [11], coincide in q. Thus, $h^0(E|_{C_H}) = 2$.

Assume that $|Z \cap H| \leq 2$. If C_H is smooth, we obtain in a similar way as above that E_{C_H} is either $\mathcal{O}_{C_H}(-2p) \oplus \mathcal{O}_{C_H}$ or $\mathcal{O}_{C_H}(-p) \oplus \mathcal{O}_{C_H}(-p)$ and thus $h^0(E|_{C_H}) \leq 1$. When H is a tangent plane section at $q \in Q$, we can

also similarly show that $h^0(E|_{C_H}) \leq 1$, except when $Z \cap H = \{x, y\}$ and $y = q$, say $x \in l_1$. In this case, we have

$$E|_{l_1} \simeq \mathcal{O}_{l_1}(1) \oplus \mathcal{O}_{l_1}(-2), \text{ and}$$
$$E|_{l_2} \simeq \mathcal{O}_{l_2} \oplus \mathcal{O}_{l_2}(-1).$$

Since $y = q$ is the intersection point of l_1 and l_2, the sub-bundles $\mathcal{O}_{l_1}(1)$ and $\mathcal{O}_{l_2}$ in (9) do not coincide at y. So $h^0(E|_{C_H}) = 1$. $\square$

Since we have $\binom{k}{3}$ hyperplanes that meet Z at 3 points and thus $S(E)$ has at least $\binom{k}{3}$ singular points. Thus we have the following statement.

Proposition 3.2. *For a Hulsbergen bundle $E \in \mathcal{M}(k)$, $S(E)$ is a hypersurface of degree $k-1$ in $\mathbb{P}_3^*$ with $\binom{k}{3}$ singular points.*

3.1. If $c_2 = 1$, then there is no stable vector bundles. In fact, it can be shown [8] that there exists a unique strictly semi-stable vector bundle $E_0 := \mathcal{O}_Q(-1,0) \oplus \mathcal{O}_Q(0,-1)$. Since $h^0(E_0) = h^1(E_0(-1,-1)) = 0$, we have $h^0(E_0|_{C_H}) = 0$ for all $H \in \mathbb{P}_3^*$. Hence, if we extend the concept of the jumping conic to semi-stable bundles, we can say that there is no jumping conic of E_0. It is consistent with the fact that $S(E_0)$ is a hypersurface of degree 0 in $\mathbb{P}_3^*$.

3.2. If $c_2 = 2$, then $S(E)$ is a hyperplane in $\mathbb{P}_3^*$. So the map S is from $\mathcal{M}(2)$ to $\mathbb{P}_3$. It was shown in [8] that S extends to an isomorphism

$$\overline{S} : \overline{\mathcal{M}}(2) \to \mathbb{P}_3,$$

where $\overline{\mathcal{M}}(2)$ is the compactification of $\mathcal{M}(2)$ in the sense of Gieseker [6], whose boundary consists of non-locally free sheaves with the same numeric invariants. In fact, for $E \in \overline{\mathcal{M}}(2)$, we have $h^0(E(1,1)) = 3$ and can define a morphism from $\mathbb{P}_2 \simeq \mathbb{P}H^0(E(1,1))$ to the Grassmannian $Gr(1,3)$, sending a section s to the line in $\mathbb{P}_3$ containing the two zeros of s. The image of this map can be shown to be a 2-cycles of $Gr(1,3)$ corresponding to the unique point in $\mathbb{P}_3$. $\overline{S}$ maps E to this uniquely determined point. Moreover, $\mathcal{M}(2)$ maps to $\mathbb{P}_3 \backslash Q$ via S and in particular, $S(E)$ determines E completely. Let Z be a 0-cycle on Q with length 2 such that the support of Z does not lie on a line in Q and consider an extension family $\mathbb{P}(Z)$ of E, admitting the following exact sequence,

$$0 \to \mathcal{O}_Q \to E(1,1) \to I_Z(1,1) \to 0.$$

Then, $\mathbb{P}(Z) \simeq \mathbb{P}_1$ is the secant line of Q passing through the support of Z. From this description, it can be easily checked that $H \in S(E)$ if and only if $E|_{C_H} \simeq \mathcal{O}_{C_H} \oplus \mathcal{O}_{C_H}(-2p)$, which is consistent with the fact that $S(E)$ is smooth.

3.3. If $c_2 = 3$, we have a map $S : \mathcal{M}(3) \to |\mathcal{O}_{\mathbb{P}_3^*}(2)| \simeq \mathbb{P}_9$, where $S(E)$ is a quadric in $\mathbb{P}_3^*$. $E(1,1)$ is fitted into the following exact sequence,

$$0 \to \mathcal{O}_Q \to E(1,1) \to I_Z(1,1) \to 0, \tag{10}$$

with a 0-cycle Z on Q with length 3. If Z is contained in a line on Q, then E contains $\mathcal{O}_Q(0,-1)$ or $\mathcal{O}_Q(-1,0)$ as a sub-bundle, contradicting to the stability of E. Thus there exists a unique hyperplane H in $\mathbb{P}_3$ containing Z.

Remark 3.3. Conversely, if Z is not contained in any line on Q, then it can be easily shown from the standard computation that any sheaf E admitting an exact sequence (10) is semi-stable. In fact, if a subscheme of length 2 of Z is contained in a line on Q, any sheaf E admitting (10) is strictly semi-stable.

Now let us consider a map

$$\eta_E : \mathbb{P}_1 \simeq \mathbb{P}H^0(E(1,1)) \to Gr(2,3) \simeq \mathbb{P}_3^*,$$

sending a section $s \in H^0(E(1,1))$ to the projective plane in $\mathbb{P}_3$ containing a 0-cycle Z in the exact sequence (10), which is obtained from s. Before proving that η_E is a constant map, we suggest a different proof of the fact that $S(E)$ is a quadric cone in $\mathbb{P}_3^*$.

Proposition 3.4. *For $E \in \mathcal{M}(3)$, $S(E)$ is a quadric cone in $\mathbb{P}_3^*$ with a vertex point.*

Proof. Let s be a section of $E(1,1)$ from which $E(1,1)$ admits an exact sequence (10) for a 0-dimensional cycle Z of length 3. Let $Z = \{z_1, z_2, z_3\}$. If H_s be a hyperplane in $\mathbb{P}_3$ containing Z, then $E|_{C_{H_s}}$ admits an exact sequence,

$$0 \to \mathcal{O}_{C_{H_s}}(p) \to E|_{C_{H_s}} \to \mathcal{O}_{C_{H_s}}(-3p) \to 0,$$

where p is a point on C_{H_s}. It splits since $H^1(\mathcal{O}_{C_{H_s}}(-6p)) = 0$. Thus $E_{C_{H_s}}$ is isomorphic to $\mathcal{O}_{C_{H_s}}(p) \oplus \mathcal{O}_{C_{H_s}}(-3p)$ and in particular, $H_s \in S(E)$. Similarly, if H contains only 2 points of Z, then $H \in S(E)$. It can be shown that $H \notin S(E)$ if H contains only 1 point of Z. Let us consider a hyperplane $H(z_1)$ in $\mathbb{P}_3^*$, whose points correspond to the hyperplanes in $\mathbb{P}_3$ containing z_1. From the previous argument, we know that the intersection of $H(z_1)$ with $S(E)$ consists of 2 straight lines whose intersection point corresponds to the hyperplane H_s. If $S(E)$ is a smooth quadric, then $H(z_1)$ is the tangent plane of $S(E)$ at H_s. Similarly, we can define $H(z_i)$, $i = 2, 3$, and they would also become the tangent plane of $S(E)$ at H_s, which is absurd. We can similarly derive a contradiction in the case when Q is a hyperplane in $\mathbb{P}_3^*$ with multiplicity 2. Let us assume that $S(E)$ consists of two hyperplanes meeting at a line l. Clearly, H_s lies in l. There are 3 lines on $S(E)$ corresponding to the hyperplanes containing 2 points of Z and they are exactly the intersection of $H(z_i)$'s. Hence there is a hyperplane of $S(E)$ that contains two intersecting lines of $H(z_i)$'s. It is impossible since the two intersecting lines of $H(z_i)$ with $S(E)$ lie on different components of $S(E)$. Thus Q is a quadric cone with a vertex point. ∎

Corollary 3.5. *For $E \in \mathcal{M}(3)$, the map η_E is a constant map to the vertex point of $S(E)$.*

Proof. Using the notation in the proof of the preceding proposition, the planes $H(z_i)$ meet with $S(E)$ at two different lines. The only possibility is that H_s is the vertex point of $S(E)$. Now we get the assertion since this argument is valid for all sections of $E(1,1)$. □

Remark 3.6. The hyperplane H corresponding to the vertex point of $S(E)$ is the unique hyperplane for which $E|_{C_H}$ is isomorphic to $\mathcal{O}_{C_H}(-3p) \oplus \mathcal{O}_{C_H}(p)$, where p is a point on Q. For the other hyperplanes in $S(E)$, $E|_{C_H}$ become $\mathcal{O}_{C_H}(-2p) \oplus \mathcal{O}_{C_H}$.

By sending $E \in \mathcal{M}(3)$ to the vertex point of $S(E)$, we can define a map

$$\Lambda^* : \mathcal{M}(3) \to \mathbb{P}_3^*.$$

Let p be a point in $\mathbb{P}_3^* \backslash Q^*$, where Q^* is the dual of Q, whose points correspond to the tangent planes of Q. We can pick a stable vector bundle E fitted into the exact sequence (10) for a 0-cycle Z of length 3 whose support lies in the hyperplane section corresponding to p. Then E maps to the point p via Λ^*. In the case when $p \in Q^*$, we can also choose a 0-cycle Z for which there exists a stable vector bundle E mapping to p. Thus Λ^* is surjective and its generic fibres are 4-dimensional.

Now let us consider the determinant map

$$\lambda_E : \wedge^2 H^0(E(1,1)) \to H^0(\mathcal{O}_Q(1,1)).$$

Since $h^0(E(1,1)) = 2$, the dimension of the domain is 1-dimensional.

Lemma 3.7. *λ_E is injective.*

Proof. We follow the argument in the proof of the lemma (6.6) in [14]. Let s_1, s_2 be two linearly independent sections of $E(1,1)$. Assume that $s_1 \wedge s_2$ maps to 0 via λ_E. It would generate a line subbundle L of $E(1,1)$ with $h^0(L) = 2$. The only choices for L is $\mathcal{O}_Q(0,1)$ and $\mathcal{O}_Q(1,0)$, and both contradict the stability of E. □

Let us define q_E to be the point in $\mathbb{P}_3^* \simeq \mathbb{P}H^0(\mathcal{O}_Q(1,1))$ corresponding to the image of λ_E. Since $E(1,1)$ is fitted into the exact sequence (10), $H^0(E(1,1))$ can be considered as the direct sum of $H^0(\mathcal{O}_Q)$ and $H^0(I_Z(1,1))$, so $\wedge^2 H^0(E(1,1))$ is isomorphic to $H^0(I_Z(1,1))$. From the long exact sequence of cohomology of the exact sequence,

$$0 \to I_Z(1,1) \to \mathcal{O}_Q(1,1) \to \mathcal{O}_Z \to 0,$$

$H^0(I_Z(1,1))$ is embedded into $H^0(\mathcal{O}_Q(1,1))$. This embedding is determined by the injection of $H^0(\mathcal{O}_Z)^*$ into $H^0(\mathcal{O}_Q(1,1))^*$, i.e. the hyperplane in $\mathbb{P}_3$ containing Z. We know from the preceding corollary that this hyperplane is independent on the sections of $E(1,1)$. Thus the embedding of $H^0(I_Z(1,1))$ into $H^0(\mathcal{O}_Q(1,1))$ is independent on Z and it would give the same map as λ_E. As a quick consequence of this argument, we obtain that the image

of λ_E corresponds to the unique hyperplane in $\mathbb{P}_3$ containing Z. In other words, we obtain the following statement.

Proposition 3.8. *q_E is the vertex point of $S(E)$.*

Remark 3.9. Let f_Q be the polar map from $\mathbb{P}_3$ to $\mathbb{P}_3^*$ given by

$$[x_0, \cdots, x_3] \mapsto [\frac{\partial f}{\partial t_0}(x), \cdots, \frac{\partial f}{\partial t_3}(x)], \tag{11}$$

where f is the homogeneous polynomial of degree 2 defining Q. Then we have a surjective map from $\mathcal{M}(3)$ to $\mathbb{P}_3$,

$$\Lambda := f_Q^{-1} \circ \Lambda^* : \mathcal{M}(3) \to \mathbb{P}_3.$$

For $E \in \mathcal{M}(3)$, let H_E be the hyperplane of $\mathbb{P}_3$ corresponding to q_E. Note that $C_{H_E} = H_E \cap Q$ is the set of points $p \in Q$ for which $\Lambda(E)$ is contained in the tangent plane of Q at p. Thus we can define the map Λ by sending E to the intersection point of the tangent planes at the support of Z in the exact sequence (10), which is independent on the choice of a section of $E(1,1)$.

Recall that the set of singular quadrics in $\mathbb{P}_3^*$ is the discriminant hypersurface $\mathcal{D}_2$ in $\mathbb{P}_9$ defined by the equation $\det(\mathcal{A}) = 0$, where $\mathcal{A}$ is a symmetric 4×4-matrix. By differentiating, we know that the singular points of $\mathcal{D}_2$ are defined by the determinants of 3×3-minors of $\mathcal{A}$, i.e. the singular points of $\mathcal{D}_2$ correspond to the singular quadrics of rank ≤ 2. Let $\mathcal{D}_2^0$ be the smooth part of $\mathcal{D}_2$. Then we have the following picture,

$$\begin{tikzcd} \mathcal{M}(3) \arrow[r, "S"] \arrow[dr, "\Lambda^*"'] & \mathcal{D}_2^0 \arrow[d] \\ & \mathbb{P}_3^*, \end{tikzcd} \tag{12}$$

where $\mathcal{D}_2^0$ is an open Zariski subset of a quartic hypersurface $\mathcal{D}_2$ of $\mathbb{P}_9$ and the vertical map sends a singular quadric of rank 3 to its vertex point.

Let $E \in (\Lambda^*)^{-1}(q_E)$ with $q_E \notin Q^*$. Thus H_E is not a tangent plane of Q and so C_{H_E} is a smooth conic on H_E. Let $\mathbb{P}_2^*$ be the image of H_E via the polar map f_Q, which is a hyperplane of $\mathbb{P}_3^*$, not containing q_E. Then $\mathbb{P}_2^*$ contains the dual conic $C_{H_E}^*$ of C_{H_E} via $f_Q|_{H_E}$. Let π_{q_E} be the projection map from $\mathbb{P}_3^*$ to $\mathbb{P}_2^*$ at q_E. Then we can assign a smooth conic $C(E) := \pi_{q_E}(S(E)) \subset \mathbb{P}_2^*$ to E, i.e. we have a map

$$\pi_{q_E} : (\Lambda^*)^{-1}(q_E) \to |\mathcal{O}_{\mathbb{P}_2}^*(2)| \simeq \mathbb{P}_5.$$

Clearly, $C(E) \neq C_{H_E}^*$.

Let us fix a general hyperplane H of $\mathbb{P}_3$. For a 0-cycle Z with length 3 contained in $C_H \simeq \mathbb{P}_1$, we can consider an extension space $\mathbb{P}(Z) := \mathbb{P}\operatorname{Ext}^1(I_Z(1,1), \mathcal{O}_Q) \simeq \mathbb{P}_2$. Note that the Hilbert scheme parametrizing 0-cycles on C_H with length 3, $\mathbb{P}_1^{[3]}$, is isomorphic to $\mathbb{P}_3$. Let us define

$$\mathcal{U} := R^1 p_{1*}(\mathcal{I} \otimes p_2{}^* \mathcal{O}_Q(-1,-1)),$$

where p_1, p_2 are the projections from $\mathbb{P}_3 \times Q$ to each factors and $\mathcal{I}$ is the universal ideal sheaf of $\mathbb{P}_3 \times Q$. We can easily find that $\mathcal{U}$ is a vector bundle on $\mathbb{P}_3$ of rank 3 and the fibre of $\mathbb{P}(\mathcal{U}^*)$ at $Z \in \mathbb{P}_3$ is $\mathbb{P}(Z)$. Then we have a rational map from $\mathbb{P}(\mathcal{U}^*)$ to $\mathcal{M}(3)_q := (\Lambda^*)^{-1}(q)$, and eventually to $\mathbb{P}_5$ after the composition with π_q, where q corresponds to H. In particular, the dimension of the image of $\mathbb{P}(\mathcal{U}^*)$ is less than 5 since the dimension of $\mathcal{M}(3)_q$ is 4.

$$
\begin{tikzcd}
\mathbb{P}(\mathcal{U}^*) \arrow[rr] \arrow[dr] & & \mathbb{P}_5 \\
& \mathcal{M}(3)_q \arrow[ur] &
\end{tikzcd}
\tag{13}
$$

For a general 0-cycle $Z = \{z_1, z_2, z_3\}$ on C_H, let $p_{ij} \in \mathbb{P}_2^*$ be the point corresponding to the line containing z_i and z_j. The conic $C(E)$ contains p_{ij} and so the image of $\mathbb{P}(Z)$ is contained in the projective plane in $\mathbb{P}_5$ parametrizing all the conics passing through three points p_{ij}. Let $Z^* = \{z_1^*, z_2^*, z_3^*\}$ be the dual lines on $\mathbb{P}_2^*$ of Z, then p_{ij} is the intersection point of z_i^* and z_j^*. If we choose linear forms $0 \neq Z_i \in H^0(\mathcal{O}_{\mathbb{P}_2^*}(1))$ which vanish on z_i^*, then from the previous statement, $\pi_q \circ S$ is defined by

$$
\begin{aligned}
\pi_q \circ S : \mathbb{P}(Z) &\to |\mathcal{O}_{\mathbb{P}_2^*}(2)| \\
(c_1, c_2, c_3) &\mapsto f_1 Z_2 Z_3 + f_2 Z_1 Z_3 + f_3 Z_1 Z_2,
\end{aligned}
$$

where (c_1, c_2, c_3) is the coordinates from the identification of $\mathbb{P}(Z)$ with $\mathbb{P}H^0(\mathcal{O}_Z)^*$ and f_i's are homogeneous polynomials of c_j's.

Proposition 3.10. *For a general 0-cycle Z, the map $\pi_q \circ S$ from $\mathbb{P}(Z)$ to $\mathbb{P}_5$ sending E to $\pi_q(S(E))$, is a linear embedding.*

Proof. From the previous argument, it is enough to check that f_i's are linearly independent linear polynomials. In fact we can prove that $f_i \equiv c_i$ for all i.

Recall that $\mathbf{I}$ is the incidence variety in $Q \times \mathbb{P}_3^*$ with the projections π_1 and π_2. Then we have an isomorphism,

$$h : \mathcal{O}_{\mathbb{P}_3^*} \to \pi_{2*}\pi_1^* I_Z((0,0),3),$$

given by the multiplication with $Z_1Z_2Z_3$. Here, $\mathcal{O}_{\mathbf{I}}((a,b),c)$ is the sheaf $\pi_1^*\mathcal{O}_Q(a,b) \otimes \pi_2^*\mathcal{O}_{\mathbb{P}_3^*}(c)$ on $\mathbf{I}$. Note that $\pi_{2*}\pi_1^* I_Z$ is the ideal sheaf of functions on $\mathbb{P}_3^*$, vanishing on the lines z_i^*. From the canonical homomorphisms,

$$
\begin{aligned}
\mathrm{Ext}^1(I_Z(1,1), \mathcal{O}_Q) &\to \mathrm{Ext}^1(\pi_1^* I_Z(1,1), \mathcal{O}_{\mathbf{I}}) \\
&\to \mathrm{Ext}^1(\pi_1^* I_Z((0,0),3), \mathcal{O}_{\mathbf{I}}((-1,-1),3)),
\end{aligned}
$$

we can assign to an element $\varepsilon \in \mathrm{Ext}^1(I_Z(1,1), \mathcal{O}_Q)$, an extension

$$0 \to \mathcal{O}_{\mathbf{I}}((-1,-1),3) \to \pi_1^* E((0,0),3) \to \pi_1^* I_Z((0,0),3) \to 0. \tag{14}$$

From the long exact sequence of cohomology of (14), we obtain

$$H^0(\mathcal{O}_{\mathbb{P}_3}^*) \to H^0(\pi_1^* I_Z((0,0),3)) \to H^1(\mathcal{O}_{\mathbf{I}}((-1,-1),3)) \simeq H^0(\mathcal{O}_{\mathbb{P}_3^*}(2)),$$

and let $\pi(\varepsilon)$ be the image of $1 \in H^0(\mathcal{O}_{\mathbb{P}_3^*})$. Then we can define a homomorphism

$$\pi : \mathrm{Ext}^1(I_Z(1,1), \mathcal{O}_Q) \to H^0(\mathcal{O}_{\mathbb{P}_3^*}(2)), \tag{15}$$

by sending ε to $\pi(\varepsilon)$.

From the inclusion $I_{z_i} \hookrightarrow I_Z$, we have a natural injection from

$$\mathrm{Ext}^1(I_{z_i}(1,1), \mathcal{O}_Q) \simeq \mathbb{C} \hookrightarrow \mathrm{Ext}^1(I_Z(1,1), \mathcal{O}_Q)$$

whose image is $H^0(\mathcal{O}_{z_i})^*$. It can be easily checked that any element in the image is mapped to $H^0(\mathcal{O}_{\mathbb{P}_3^*}(2))$ by the multiplication with $(Z_1Z_2Z_3)/Z_i$. Thus π is defined by sending (c_1, c_2, c_3) to $c_1Z_2Z_3 + c_2Z_1Z_3 + c_3Z_1Z_2$.

When we take the direct image of (14), then we obtain

$$\begin{aligned} \pi_{2*}\pi_1^*E((0,0),3) \to \pi_{2*}\pi_1^*I_Z((0,0),3) \to R^1\pi_{2*}\mathcal{O}_{\mathbf{I}}((-1,-1),3) \\ \to R^1\pi_{2*}\pi_1^*E((0,0),3) \to R^1\pi_{2*}\pi_1^*I_Z((0,0),3) \to 0. \end{aligned}$$

Note that $\pi_{2*}\pi_1^*I_Z((0,0),3) \simeq \mathcal{O}_{\mathbb{P}_3^*}$, $R^1\pi_{2*}\mathcal{O}_{\mathbf{I}}((-1,-1),3) \simeq \mathcal{O}_{\mathbb{P}_3^*}(2)$ and the second map in the sequence above, is given by the multiplication with $\pi(\varepsilon)$. As an analogue of the result in [9], we can easily check that $R^1\pi_{2*}\pi_1^*E((0,0),3)$ is isomorphic to $\vartheta_E(4)$ and its support is $S(E)$. On the other hand, the support of $R^1\pi_{2*}\pi_1^*I_Z((0,0),3)$ is contained in $\{p_{ij}\}$ and thus the support of $S(E)$ is same as the support of $\{\pi(\varepsilon) = 0\}$. Because of the same degree, they are the same. $\square$

Remark 3.11. Using the argument as in the similar statement on the projective plane in [10], we can prove that a sheaf $E \in \mathbb{P}(Z)$ with the coordinates (c_1, c_2, c_3) is locally free if and only if $c_i \neq 0$ for all i. Thus from the proof of the preceding proposition, we can observe that the conic corresponding to the image of E is smooth if and only if E is locally free. Note that the secant variety V_3 of the Veronese surface in $\mathbb{P}_5$ is a cubic hypersurface. The intersection of the image of $\mathbb{P}(Z)$ with V_3 are the 3 lines, which are the image of non-locally free sheaves in $\mathbb{P}(Z)$.

We can see that the same statement holds for arbitrary hyperplane section $H \in \mathbb{P}_3^*$. If $H \in Q^*$, Q^* the dual conic of Q, then $C_H = l_1 \cup l_2$. Because of the stability condition, our 0-cycles of length 3 associated to E with $\Lambda^*(E) \in Q^*$ cannot have its support only on l_1 nor l_2. So the family of 0-cycles we consider, is isomorphic to the two copies of $\mathbb{P}_1^{[2]} \times \mathbb{P}_1$. Let us denote

$$\mathcal{M}(3) = \mathcal{M}^0(3) \coprod \mathcal{M}^1(3) \coprod \mathcal{M}^2(3),$$

where $\mathcal{M}^0(3) = (\Lambda^*)^{-1}(\mathbb{P}_3^* \backslash Q^*)$ and $\mathcal{M}^i(3)$'s are the two irreducible components of $(\Lambda^*)^{-1}(Q^*)$ whose 0-cycles have two points of its support on the ruling equivalent to l_i.

First let us assume that $H \notin Q^*$. Let $V \subset \mathbb{P}_5$ be the image of $\mathbb{P}(\mathcal{U}^*)$ and $v \in V$ be a general point in V. Then there exists three points z_i's on C_H and c_i's for which we have $v = c_1Z_1 + c_2Z_2 + c_3Z_3$. Since $z_i \in C_H$, the lines

Z_i's are tangent to the dual conic C_H^*, i.e. Z_i's is a circumscribed triangle around C_H^*. Note that Z_i's is a inscribed triangle in v. Thus V is the closure of the family of conics Poncelet related to C_H^* (see section 2 in [5]). From the classical result, V is a hypersurface in $\mathbb{P}_5$ and the generic fibre of the map $\mathbb{P}(\mathcal{U}^*) \to V$ is isomorphic to $\mathbb{P}_1$. In fact, from the remark (2.2.3) in [5], V is isomorphic to a hypersurface of degree 4, H_4 in the space of conics, given by the condition $c_2^2 - c_1c_3 = 0$, where

$$\det(A - tI_3) = (-t)^3 + c_1(-t)^2 + c_2(-t) + c_3,$$

is the characteristic polynomial of a symmetric matrix A defining a conic.

Let $E \in \mathcal{M}^1(3)$ and so $H \in Q^*$. If we define V as before and let $v \in V$ be a general conic, then v pass through the dual point $p_1 \in \mathbb{P}_2$ of l_1. Let us fix a conic v passing through p_1. If we choose $q_1 \in v$ not equal to p_1, then consider a line l passing through q_1 and the dual point p_2 of l_2. Let q_2 be the other intersection point of l with v. Then the dual points corresponding to the lines $\overline{p_1q_1}, \overline{q_1q_2}, \overline{q_2p_1}$ is a 0-cycle Z mapping to v. It depends on the choice of q_1. Thus, V is isomorphic to a hyperplane in $\mathbb{P}_5$ and the generic fibre of the map from $\mathbb{P}(\mathcal{U}^*)$ is again isomorphic to $\mathbb{P}_1$. We have the same argument for $\mathcal{M}^2(3)$.

As a direct consequence, $\mathcal{M}(3)_q$ is isomorphic to an open Zariski subset of a hyperplane in $\mathbb{P}_5$ and thus we have the following proposition.

Proposition 3.12. *$\mathcal{M}(3)$ admits a fibration over $\mathbb{P}_3^*$ whose fibre over $H \in \mathbb{P}_3^*$ is isomorphic to*

(1) *an open Zariski subset $H_4 \cap (\mathbb{P}_5 \backslash V_3)$ of a H_4, where V_3 is the secant variety of the Veronese surface $S \subset \mathbb{P}_5$ and H_4 is a hypersurface of degree 4 consisting of conics Poncelet related to $Q \cap H$, if $H \in \mathbb{P}_3^* \backslash Q^*$;*
(2) *the union of two varieties $H_i \cap (\mathbb{P}_5 \backslash V_3)$, $i = 1, 2$, where H_i is the hyperplane in the space of conics which pass through a point p_i dual to the line $l_i \subset H$, where $Q \cap H = l_1 + l_2$, if $H \in Q^*$.*

Remark 3.13. In fact, we can obtain differently the old result of Darboux on the Poncelet related conics in the case of triangles. We know that we have $\dim V \le \dim \mathcal{M}(3)_q = 4$. Assume that C_H is a smooth conic. Let $\triangle_2$ be the subscheme of $C_H^{[3]}$ whose points are 0-cycles with at most 2 points as their supports. Similarly, we can define $\triangle_3 \subset \triangle_2$. Let $Z \in \triangle_2$, say $Z = \{x, x, y\}$. The map $\mathbb{P}(Z) \to \mathbb{P}_5$ is naturally defined by sending (c_1, c_2, c_3) to $(c_1 + c_2)XY + c_3X^2$. From this observation, the image of $\mathbb{P}_2$-bundle over δ_3 is $C_H \subset \mathbb{P}_5$ mapped by $|\mathcal{O}_{C_H}(2)|$. For $Z = \{x, x, y\}$, $\mathbb{P}(Z)$ is mapped to the line passing through X^2 and XY. When Y is moving along C_H, this line covers a projective plane $\mathbb{P}_2(x)$ passing through the point $X^2 \in C_H \subset \mathbb{P}_5$. Let D be the union of such projective planes over x moving along C_H. In particular, D is a subvariety of V with dimension 3 and all the non-locally free sheaves in $\mathbb{P}(\mathcal{U}^*)$ map to D. Also we have

$$V_3 \cap V = D,$$

where V_3 is the secant variety of the Veronese surface in $\mathbb{P}_5$. It also implies that V is a subvariety of $\mathbb{P}_5$ with dimension 4.

Let us consider a fibre of the map $\mathbb{P}(\mathcal{U}^*) \to V$ over XY with $x, y \in C_H$. The image of the closure of this fibre via the projection to $C_H^{[3]}$ is isomorphic to $\mathbb{P}_1$, parametrizing 0-cycles whose supports contain x and y. In fact, there exists a unique component of the closure of the fibre, mapping to $\mathbb{P}_1 \subset C_H^{[3]}$. It implies that the closure of the fibre over a generic conic v in V is isomorphic to $\mathbb{P}_1$ since there exists at most 1 point in $\mathbb{P}(Z)$ that maps to v.

In particular, the map $S : \mathcal{M}(k) \to \mathbb{P}_N$ is an isomorphism to its image, when $k \leq 3$.

Theorem 3.14. *The set of jumping conics of $E \in \mathcal{M}(k)$, determines E uniquely when $k \leq 3$.*

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